Uniform approximation of the Cox-Ingersoll-Ross process

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Abstract

The Doss-Sussmann (DS) approach is used for uniform simulation of the Cox-Ingersoll-Ross (CIR) process. The DS formalism allows to express trajectories of the CIR process through solutions of some ordinary differential equation (ODE) depending on realizations of a Wiener process involved. By simulating the first-passage times of the increments of the Wiener process to the boundary of an interval and solving the ODE, we uniformly approximate the trajectories of the CIR process. In this respect special attention is payed to simulation of trajectories near zero. From a conceptual point of view the proposed method gives a better quality of approximation (from a path-wise point of view) than standard, or even exact simulation of the SDE at some discrete time grid.

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1 Introduction

The Cox-Ingersoll-Ross process $V(t)$ is determined by the following stochastic differential equation (SDE)

$$dV(t) = k(\lambda - V(t))dt + \sigma\sqrt{V}dw(t),\ V(t_0) = V_0, \tag{1}$$

where $k,\ \lambda,\ \sigma$ are positive constants, and w is a scalar Brownian motion. Due to [6] this process has become very popular in financial mathematical applications. The CIR process is used in particular as volatility process in the Heston model [13]. It is known ([14], [15]) that for $V_0 > 0$ there exists a unique strong solution $V_{t_0,V_0}(t)$ of (1) for all $t \geq t_0 \geq 0$. The CIR process $V(t) = V_{t_0,V_0}(t)$ is positive in the case $2k\lambda \geq \sigma^2$ and nonnegative in the case $2k\lambda < \sigma^2$. Moreover, in the last case the origin is a reflecting boundary.

As a matter of fact, (1) does not satisfy the global Lipschitz assumption. The difficulties arising in a simulation method for (1) are connected with this fact and with the natural requirement of preserving nonnegative approximations. A lot of approximation

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methods for the CIR processes are proposed. For an extensive list of articles on this subject we refer to [3] and [7]. Besides [3] and [7] we also refer to [1, 2, 11, 12], where a number of discretization schemes for the CIR process can be found. Further we note that in [17] a weakly convergent fully implicit method is implemented for the Heston model. Exact simulation of (1) is considered in [5, 9] (see [3] as well).

In this paper, we consider uniform pathwise approximation of $V(t)$ on an interval $[t_0, t_0 + T]$ using the Doss-Sussmann transformation ([8], [20], [19]) which allows for expressing any trajectory of $V(t)$ by the solution of some ordinary differential equation that depends on the realization of $w(t)$. The approximation $\overline{V}(t)$ will be uniform in the sense that the path-wise error will be uniformly bounded, i.e.

$$\sup_{t_0 \leq t \leq t_0+T} \left|\overline{V}(t) - V(t)\right| \leq r \quad \text{almost surely,} \tag{2}$$

where $r > 0$ is fixed in advance. In fact, by simulating the first-passage times of the increments of the Wiener process to the boundary of an interval and solving this ODE, we approximately construct a generic trajectory of $V(t)$. Such kind of simulation is more simple than the one proposed in [5] and moreover has the advantage of uniform nature. Let us consider the simulation of a standard Brownian motion W on a fixed time grid

$$t_0, t_i, ..., t_n = T.$$

Although W may be even exactly simulated at the grid points, the usual piecewise linear interpolation

$$\overline{W}(t) = \frac{t_{i+1} - t}{t_{i+1} - t_i} W(t_i) + \frac{t - t_i}{t_{i+1} - t_i} W(t_{i+1})$$

is not uniform in the sense of (2). Put differently, for any (large) positive number A, there is always a positive probability (though possibly small) that

$$\sup_{t_0 \leq t \leq t_0+T} \left|\overline{W}(t) - W(t)\right| > A.$$

Therefore, for path dependent applications for instance, such a standard, even exact, simulation method may be not desirable and a uniform method preserving (2) may be preferred.

We note that the original DS results rely on a global Lipschitz assumption that is not fulfilled for (1). We therefore have introduced the DS formalism that yields a corresponding ODE which solutions are defined on random time intervals. If V gets close to zero however, the ODE becomes intractable for numerical integration and so, for the parts of a trajectory $V(t)$, that are close to zero, we are forced to use some other (not DS) approach. For such parts we here propose a different uniform simulation method. Another restriction is connected with the condition $\alpha := \left(4k\lambda - \sigma^2\right)/8 > 0$. We note that the case $\alpha > 0$ is more general than the case $2k\lambda \geq \sigma^2$ that ensures positivity of $V(t)$, and stress that in the literature virtually all convergence proofs for methods for numerical integration of (1) are based on the assumption $2k\lambda \geq \sigma^2$. We expect that the results here obtained for $\alpha > 0$ can be extended to the case where $\alpha \leq 0$, however in a highly nontrivial way. Therefore, the case $\alpha \leq 0$ will be considered in a subsequent work.

The next two sections are devoted to DS formalism in connection with (1) and to some auxiliary propositions. In Sections 4 and 5 we deal with the one-step approximation and the convergence of the proposed method, respectively. Section 6 is dedicated to the uniform construction of trajectories close to zero.

2 The Doss-Sussmann transformation

2.1 Due to the Doss-Sussmann approach ([8], [14], [19], [20]), the solution of (1) may be expressed in the form

$$V(t) = F(X(t), w(t)), \tag{3}$$

where $F = F(x, y)$ is some deterministic function and $X(t)$ is the solution of some ordinary differential equation depending on the part $w(s)$, $0 \le s \le t$, of the realization $w(\cdot)$ of the Wiener process $w(t)$.

Let us recall the Doss-Sussmann formalism according to [19], V.28. In [19] one consideres the Stratonovich SDE

$$dV(t) = b(V)dt + \gamma(V) \circ dw(t). \tag{4}$$

The function $F = F(x, y)$ is found from the equation

$$\frac{\partial F}{\partial y} = \gamma(F),\ F(x,0) = x, \tag{5}$$

and $X(t)$ is found from the ODE

$$\frac{dX}{dt} = \frac{1}{\partial F/\partial x(X(t), w(t))} b(F(X(t), w(t)),\ X(0) = V(0). \tag{6}$$

It turns out that application of the DS formalism after the Lamperti transformation $U(t) = \sqrt{V(t)}$ (see [7]) leads to more simple equations. The Lamperti transformation yields the following SDE with additive noise

$$dU = (\frac{\alpha}{U} - \frac{k}{2}U)dt + \frac{\sigma}{2}dw,\ U(0) = \sqrt{V(0)} > 0, \quad \text{where} \tag{7}$$

$$\alpha = \frac{4k\lambda - \sigma^2}{8}. \tag{8}$$

Let us seek the solution of (7) in the form

$$U(t) = G(Y(t), w(t)) \tag{9}$$

in accordance with (3)-(6). Because the Ito and Stratonovich forrms of equation (7) coincide, we have

$$b(U) = \frac{\alpha}{U} - \frac{k}{2}U,\ \gamma(U) = \frac{\sigma}{2}.$$

The function $G = G(y, z)$ is found from the equation

$$\frac{\partial G}{\partial z} = \frac{\sigma}{2},\ G(y,0) = y,$$

i.e.,

$$G(y, z) = y + \frac{\sigma}{2}z, \tag{10}$$

and $Y(t)$ is found from the ODE

$$\frac{dY}{dt} = \frac{\alpha}{Y + \frac{\sigma}{2}w(t)} - \frac{k}{2}(Y + \frac{\sigma}{2}w(t)),\ Y(0) = U(0) = \sqrt{V(0)} > 0. \tag{11}$$

From (9), (10), and solution of (11), we formally obtain the solution $U(t)$ of (7):

$$U(t) = Y(t) + \frac{\sigma}{2} w(t). \tag{12}$$

Hence

$$V(t) = U^2(t) = (Y(t) + \frac{\sigma}{2} w(t))^2. \tag{13}$$

2.2 Since the Doss-Sussmann results rely on a global Lipschitz assumption that is not fulfilled for (1), solution (13) has to be considered only formally. In this section we therefore give a direct proof of the following more precise result.

Proposition 1 *Let $Y(0) = U(0) = \sqrt{V(0)} > 0$. Let τ be the following stopping time:*

$$\tau := \inf\{t : V(t) = 0\}.$$

Then equation (11) has a unique solution $Y(t)$ on the interval $[0, \tau)$, the solution $U(t)$ of (7) is expressed by formula (12) on this interval, and $V(t)$ is expressed by (13).

Proof. Let $(w(t),\ V(t))$ be the solution of the SDE system

$$dw = dw(t),\ dV = k(\lambda - V)dt + \sigma\sqrt{V(t)}dw\,(t)\,,$$

which satisfies the initial conditions $w(0) = 0,\ V(0) > 0$. Then $U(t) = \sqrt{V(t)} > 0$ is a solution of (7) on the interval $[0, \tau)$. Consider the function $Y(t) = U(t) - \frac{\sigma}{2} w(t),\ 0 \le t < \tau$. Clearly, $Y(t) + \frac{\sigma}{2} w(t) > 0$ on $[0, \tau)$. Due to Ito's formula, we get

$$dY(t) = dU(t) - \frac{\sigma}{2} dw(t) = \frac{\alpha dt}{Y + \frac{\sigma}{2} w(t)} - \frac{k}{2}(Y + \frac{\sigma}{2} w(t))dt,$$

i.e., the function $U(t) - \frac{\sigma}{2} w(t)$ is a solution of (11). The uniqueness of $Y(t)$ follows from the uniqueness of $V(t)$. ■

2.3 So far we were starting at the moment $t = 0$. It is useful to consider the Doss-Sussmann transformation with an arbitrary initial time $t_0 > 0$ (which even may be a stopping time, for example, $0 \le t_0 < \tau$). In this case, we obtain instead of (11) for

$$Y = Y(t; t_0) = U(t) - \frac{\sigma}{2}(w(t) - w(t_0)) = \sqrt{V(t)} - \frac{\sigma}{2}(w(t) - w(t_0)),\ t_0 \le t < t_0 + \tau,$$

the equation

$$\frac{dY}{dt} = \frac{\alpha}{Y + \frac{\sigma}{2}(w(t) - w(t_0))} - \frac{k}{2}(Y + \frac{\sigma}{2}(w(t) - w(t_0))), \tag{14}$$
$$Y(t_0; t_0) = \sqrt{V(t_0)},\ t_0 \le t < t_0 + \tau,$$

with α given by (8). Clearly,

$$V(t) = (Y(t; t_0) + \frac{\sigma}{2}(w(t) - w(t_0)))^2,\ t_0 \le t < t_0 + \tau. \tag{15}$$

3 Auxiliary propositions

3.1 Let us consider in view of (14) solutions of the ordinary differential equations

$$\frac{dy^0}{dt} = \frac{\alpha}{y^0} - \frac{k}{2}y^0,\ y^0(t_0) = y_0 > 0,\ t \geq t_0 \geq 0, \tag{16}$$

which are given by

$$y^0(t) = y^0_{t_0,y_0}(t) = [y_0^2 e^{-k(t-t_0)} + \frac{2\alpha}{k}(1 - e^{-k(t-t_0)})]^{1/2},\ t \geq t_0. \tag{17}$$

In the case $\alpha > 0$, i.e., $4k\lambda > \sigma^2$, we have: if $y_0 > \sqrt{2\alpha/k}$ then $y^0_{t_0,y_0}(t) \downarrow \sqrt{2\alpha/k}$ as $t \to \infty$ and if $0 < y_0 < \sqrt{2\alpha/k}$ then $y^0_{t_0,y_0}(t) \uparrow \sqrt{2\alpha/k}$ as $t \to \infty$. Further $y^0(t) = \sqrt{2\alpha/k}$ is a solution of (16).

In the case $\alpha = 0$, the solution $y^0_{t_0,y_0}(t) \downarrow 0$ under $t \to \infty$ for any $y_0 > 0$. We note that the case $\alpha \geq 0$ is more general than the case $2k\lambda \geq \sigma^2$ (we recall that in the latter case $V(t) > 0,\ t \geq t_0$).

In the case $\alpha < 0$, the solution $y^0_{t_0,y_0}(t)$ is convexly decreasing under not too large y_0. It attains zero at the moment $\bar{t}$ given by

$$\bar{t} = t_0 + \frac{1}{k}\ln\frac{y_0^2 - 2\alpha/k}{-2\alpha/k} \tag{18}$$

and $y^{0\prime}_{t_0,y_0}(\bar{t}) = -\infty$.

In what follows we deal with the case

$$\alpha = \frac{4k\lambda - \sigma^2}{8} \geq 0. \tag{19}$$

3.2. Our next goal is to obtain estimates for solutions of the equation

$$\frac{dy}{dt} = \frac{\alpha}{y + \frac{\sigma}{2}\varphi(t)} - \frac{k}{2}(y + \frac{\sigma}{2}\varphi(t)),\ y(t_0) = y_0,\ t_0 \leq t \leq t_0 + \theta, \tag{20}$$

(cf. (14)) for a given continuous function $\varphi(t)$.

Lemma 2 *Let $\alpha \geq 0$. Let $y^i(t),\ i = 1, 2$, be two solutions of (20) such that $y^i(t) + \frac{\sigma}{2}\varphi(t) > 0$ on $[t_0, t_0 + \theta]$, for some θ with $0 \leq \theta \leq T$. Then*

$$\left|y^2(t) - y^1(t)\right| \leq \left|y^2(t_0) - y^1(t_0)\right|,\ t_0 \leq t \leq t_0 + \theta. \tag{21}$$

Proof. We have

$$d(y^2(t) - y^1(t))^2 = 2(y^2(t) - y^1(t)) \tag{22}$$
$$\times\left(\frac{\alpha}{y^2(t) + \frac{\sigma}{2}\varphi(t)} - \frac{k}{2}(y^2(t) + \frac{\sigma}{2}\varphi(t)) - \frac{\alpha}{y^1(t) + \frac{\sigma}{2}\varphi(t)} + \frac{k}{2}(y^1(t) + \frac{\sigma}{2}\varphi(t))\right) dt.$$

From here

$$(y^2(t) - y^1(t))^2 = (y^2(t_0) - y^1(t_0))^2$$
$$+2\int_{t_0}^{t}[-\alpha\frac{(y^2(s) - y^1(s))^2}{(y^1(s) + \frac{\sigma}{2}\varphi(s))(y^2(s) + \frac{\sigma}{2}\varphi(s))} - \frac{k}{2}(y^2(s) - y^1(s))^2]ds$$
$$\leq (y^2(t_0) - y^1(t_0))^2,$$

whence (21) follows. ■

Remark 3 *It is known that for $\delta > 1$ the Bessel process BES^δ has the representation*

$$Z(t) = Z(0) + \frac{\delta - 1}{2} \int_0^t \frac{1}{Z(s)} ds + W(t), \quad 0 \leq t < \infty,$$

where W is standard Brownian Motion, $Z(t) \geq 0$ a.s., and that in particular $E \int_0^t \frac{1}{Z(s)} ds < \infty$. (See [18]; for $\delta \leq 1$ the representation of BES^δ is less simple and involves the concept of local time.) From this fact it is not difficult to show that for $\alpha > 0$ the solution of (7) may be represented as

$$U(t) = U(t_0) + \int_{t_0}^t (\frac{\alpha}{U(s)} - \frac{k}{2} U(s)) ds + \frac{\sigma}{2} \left(w(t) - w(t_0) \right), \ U(0) > 0, \quad t_0 \leq t < \infty.$$

Thus, with $Y(t) = U(t) - \frac{\sigma}{2} \left(w(t) - w(t_0) \right)$, it holds that

$$Y(t) = Y(t_0) + \int_{t_0}^t \left(\frac{\alpha}{Y(s) + \frac{\sigma}{2} \left(w(s) - w(t_0) \right)} - \frac{k}{2} \left(Y(s) + \frac{\sigma}{2} \left(w(s) - w(t_0) \right) \right) \right) ds,$$

for $Y(0) = U(0) > 0$, $0 \leq t < \infty$, and that in particular Y is continuous and of bounded variation. From this it follows that (22) holds for $t_0 \leq t \leq t_0 + T$ when $\alpha > 0$ and $\varphi(t) = w(t) - w(t_0)$ is an arbitrary Brownian trajectory, and then inequality (21) in Lemma 2 goes through for $\theta = T$.

3.3 Now consider (20) for a continuous function φ satisfying

$$|\varphi(t)| \leq r, \ t_0 \leq t \leq t_0 + \theta \leq t_0 + T, \tag{23}$$

for some $r > 0$ and $0 \leq \theta \leq T$. Along with (16), (20) with (23), we further consider the equations

$$\frac{dy}{dt} = \frac{\alpha}{y + \frac{\sigma}{2} r} - \frac{k}{2} (y + \frac{\sigma}{2} r), \ y(t_0) = y_0, \tag{24}$$

$$\frac{dy}{dt} = \frac{\alpha}{y - \frac{\sigma}{2} r} - \frac{k}{2} (y - \frac{\sigma}{2} r), \ y(t_0) = y_0. \tag{25}$$

Let us assume that $y_0 \geq \sigma r > 0$, and consider an $\eta > 0$, to be specified below, that satisfies

$$y_0 \geq \eta \geq \sigma r > 0. \tag{26}$$

The solutions of (16), (20) with (23), (24), and (25) are denoted by $y^0(t)$, $y(t)$, $y^-(t)$, and $y^+(t)$, respectively, where $y^0(t)$ is given by (17). By using (17) we derive straightforwardly that

$$y^-(t) = [(y_0 + \frac{\sigma}{2} r)^2 e^{-k(t - t_0)} + \frac{2\alpha}{k} (1 - e^{-k(t - t_0)})]^{1/2} - \frac{\sigma}{2} r, \ t_0 \leq t \leq t_0 + \theta, \tag{27}$$

$$y^+(t) = [(y_0 - \frac{\sigma}{2} r)^2 e^{-k(t - t_0)} + \frac{2\alpha}{k} (1 - e^{-k(t - t_0)})]^{1/2} + \frac{\sigma}{2} r, \ t_0 \leq t \leq t_0 + \theta. \tag{28}$$

Note that $y^-(t) + \sigma r/2 > 0$ and $y^+(t) > \sigma r/2$, $t_0 \leq t \leq t_0 + \theta$. Due to the comparison theorem for ODEs (see, e.g., [10], Ch. 3), the inequality

$$\frac{\alpha}{y + \frac{\sigma}{2} r} - \frac{k}{2} (y + \frac{\sigma}{2} r) \leq \frac{\alpha}{y + \frac{\sigma}{2} \varphi(t)} - \frac{k}{2} (y + \frac{\sigma}{2} \varphi(t)) \leq \frac{\alpha}{y - \frac{\sigma}{2} r} - \frac{k}{2} (y - \frac{\sigma}{2} r),$$

which is fulfilled in view of (23) for $y > \sigma r/2$, then implies that

$$y^-(t) \le y(t) \le y^+(t),\ t_0 \le t \le t_0 + \theta. \tag{29}$$

The same inequality holds for $y(t)$ replaced by $y^0(t)$. We thus get

$$\left|y(t) - y^0(t)\right| \le y^+(t) - y^-(t),\ t_0 \le t \le t_0 + \theta. \tag{30}$$

Proposition 4 *Let* $\alpha = \dfrac{4k\lambda - \sigma^2}{8} \ge 0$, *the inequalities (23) and (26) be fulfilled for a fixed* $\eta > 0$, *and let* $\theta \le T$. *We then have*

$$\left|y(t) - y^0(t)\right| \le Cr(t - t_0) \le Cr\theta,\ t_0 \le t \le t_0 + \theta, \quad \textit{with} \tag{31}$$
$$C = \frac{\sigma k}{2} + \frac{4\alpha\sigma}{3\eta^2} e^{\frac{k}{2}T}.$$

In particular, C *is independent of* $t_0,\ y_0,$ *and* r *(provided (26) holds).*

Proof. We estimate the difference $y^+(t) - y^-(t)$. It holds

$$\begin{aligned} y^+(t) = z^-(t) + \frac{\sigma}{2}r,\ y^-(t) = z^+(t) - \frac{\sigma}{2}r, \\ y^+(t) - y^-(t) = \sigma r - (z^+(t) - z^-(t)), \end{aligned} \tag{32}$$

where

$$z^\pm(t) = [(y_0 \pm \frac{\sigma}{2}r)^2 e^{-k(t-t_0)} + \frac{2\alpha}{k}(1 - e^{-k(t-t_0)})]^{1/2}.$$

Further,

$$z^+(t) - z^-(t) = \frac{(z^+(t))^2 - (z^-(t))^2}{z^+(t) + z^-(t)} = \frac{2y_0\sigma r e^{-k(t-t_0)}}{z^+(t) + z^-(t)}. \tag{33}$$

Using the inequality $(a^2 + b)^{1/2} \le a + b/2a$ for any $a > 0$ and $b \ge 0$, we get

$$\begin{aligned} z^+(t) \le (y_0 + \frac{\sigma}{2}r)e^{-\frac{k}{2}(t-t_0)} + \frac{\alpha}{k}\frac{(1 - e^{-k(t-t_0)})}{(y_0 + \frac{\sigma}{2}r)e^{-\frac{k}{2}(t-t_0)}}, \\ z^-(t) \le (y_0 - \frac{\sigma}{2}r)e^{-\frac{k}{2}(t-t_0)} + \frac{\alpha}{k}\frac{(1 - e^{-k(t-t_0)})}{(y_0 - \frac{\sigma}{2}r)e^{-\frac{k}{2}(t-t_0)}}, \end{aligned}$$

whence

$$z^+(t) + z^-(t) \le 2y_0 e^{-\frac{k}{2}(t-t_0)} + \frac{\alpha}{k}\frac{(1 - e^{-k(t-t_0)})}{e^{-\frac{k}{2}(t-t_0)}}\frac{2y_0}{(y_0^2 - \frac{\sigma^2}{4}r^2)}.$$

Therefore

$$\frac{1}{z^+(t) + z^-(t)} \ge \frac{1}{2y_0 e^{-\frac{k}{2}(t-t_0)}}\left(1 - \frac{\alpha}{k(y_0^2 - \frac{\sigma^2}{4}r^2)}(e^{k(t-t_0)} - 1)\right).$$

From (33) we have that

$$z^+(t) - z^-(t) \geq \sigma r e^{-\frac{k}{2}(t-t_0)} \left(1 - \frac{\alpha}{k(y_0^2 - \frac{\sigma^2}{4} r^2)} (e^{k(t-t_0)} - 1) \right)$$

and so due to (32) we get

$$0 \leq y^+(t) - y^-(t) \leq \sigma r (1 - e^{-\frac{k}{2}(t-t_0)}) + \frac{\alpha \sigma r}{k(y_0^2 - \frac{\sigma^2}{4} r^2)} (e^{\frac{k}{2}(t-t_0)} - e^{-\frac{k}{2}(t-t_0)}).$$

Since $1 - e^{-q\vartheta} \leq q\vartheta$ for any $q \geq 0,\ \vartheta \geq 0$, and $y_0^2 - \dfrac{\sigma^2}{4} r^2 \geq \dfrac{3}{4}\eta^2$ under (26), we obtain

$$0 \leq y^+(t) - y^-(t) \leq \frac{\sigma r k}{2}(t - t_0) + \frac{4\alpha\sigma r}{3k\eta^2} e^{\frac{k}{2}(t-t_0)} k(t - t_0).$$

From this and (30), (31) follows with $C = \dfrac{\sigma k}{2} + \dfrac{4\alpha\sigma}{3\eta^2} e^{\frac{k}{2}T}$. ∎

Corollary 5 *Under the assumptions of Proposition 4, we get by taking* $\eta = y_0$*,*

$$\begin{aligned} \left| y(t) - y^0(t) \right| &\leq \left(\frac{\sigma k}{2} + \frac{4\alpha\sigma}{3y_0^2} e^{\frac{k}{2}T} \right) r\theta, \\ &:= \left(D_1 + \frac{D_2}{y_0^2} \right) r\theta, \qquad t_0 \leq t \leq t_0 + \theta, \end{aligned}$$

where D_1 and D_2 only depend on the parameters of the CIR process under consideration and the time horizon T.

4 One-step approximation

Let us suppose that for $t_m,\ t_0 \leq t_m < t_0 + T$, $V(t_m)$ is known exactly. In fact, t_m may be considered as a realization of a certain stopping time. Consider $Y = Y(t; t_m)$ on some interval $[t_m, t_m + \theta_m]$ with $y_m := Y(t_m; t_m) = \sqrt{V(t_m)}$, given by the ODE (cf. (14)),

$$\begin{aligned} \frac{dY}{dt} &= \frac{\alpha}{Y + \frac{\sigma}{2}(w(t) - w(t_m))} - \frac{k}{2}(Y + \frac{\sigma}{2}(w(t) - w(t_m))), \qquad (34) \\ Y(t_m; t_m) &= \sqrt{V(t_m)},\ t_m \leq t \leq t_m + \theta_m. \end{aligned}$$

Assume that

$$y_m = \sqrt{V(t_m)} \geq \sigma r. \qquad (35)$$

Due to (15), the solution $V(t)$ of (1) on $[t_m, t_m + \theta_m]$ is obtained via

$$\sqrt{V(t)} = Y(t; t_m) + \frac{\sigma}{2}(w(t) - w(t_m)),\ t_m \leq t \leq t_m + \theta_m. \qquad (36)$$

Though equation (34) is (just) an ODE, it is not easy to solve it numerically in a straightforward way because of the non-smoothness of $w(t)$. We are here going to construct an approximation $y^m(t)$ of $Y(t;t_m)$ via Proposition 4. To this end we simulate the point $(t_m+\theta_m, w(t_m+\theta_m)-w(t_m))$ by simulating θ_m as being the first-passage (stopping) time of the Wiener process $w(t)-w(t_m)$, $t \geq t_m$, to the boundary of the interval $[-r,r]$. So, $|w(t)-w(t_m)| \leq r$ for $t_m \leq t \leq t_m+\theta_m$ and, moreover, the random variable $w(t_m+\theta_m)-w(t_m)$, which equals either $-r$ or $+r$ with probability $1/2$, is independent of the stopping time θ_m. A method for simulating the stopping time θ_m is given in Subsection 4.1 below. Proposition 4 and Corollary 5 then yield,

$$|Y(t;t_m)-y^m(t)| \leq \left(D_1+\frac{D_2}{y_m^2}\right) r\left(t_{m+1}-t_m\right), \quad t_m \leq t \leq t_{m+1} \quad \text{with} \tag{37}$$
$$t_{m+1} := \min(t_m+\theta_m, t_0+T),$$

where $y^m(t)$ is the solution of the problem

$$\frac{dy^m}{dt} = \frac{\alpha}{y^m} - \frac{k}{2}y^m, \; y^m(t_m) = Y(t_m;t_m) = \sqrt{V(t_m)}$$

that is given by (17) with $(t_m, y_m) = (t_m, \sqrt{V(t_m)})$. We so have,

$$\sqrt{V(t)} = Y(t;t_m) + \frac{\sigma}{2}(w(t)-w(t_m)) = y^m(t) + \frac{\sigma}{2}(w(t)-w(t_m)) + \rho^m(t),$$

where due to (37),

$$|\rho^m(t)| \leq \left(D_1+\frac{D_2}{y_m^2}\right) r\left(t_{m+1}-t_m\right), \quad t_m \leq t \leq t_{m+1}. \tag{38}$$

We next introduce the one-step approximation $\sqrt{\overline{V}(t)}$ of $\sqrt{V(t)}$ on $[t_m, t_{m+1}]$ by

$$\sqrt{\overline{V}(t)} := y^m(t) + \frac{\sigma}{2}(w(t)-w(t_m)), \quad t_m \leq t \leq t_{m+1}. \tag{39}$$

Since $|w(t_{m+1})-w(t_m)| = r$ if $t_{m+1} = t_m+\theta_m < t_0+T$, and $|w(t_{m+1})-w(t_m)| \leq r$ if $t_{m+1} = t_0+T$, the one-step approximation (39) for $t = t_{m+1}$ is given by

$$\sqrt{\overline{V}(t_{m+1})} := y^m(t_{m+1}) + \frac{\sigma}{2}(w(t_{m+1})-w(t_m)) = \tag{40}$$
$$y^m(t_{m+1}) + \frac{\sigma}{2}\cdot\begin{cases} r\xi_m & \text{with } P(\xi_m=\pm1)=1/2, \text{ if } t_{m+1}=t_m+\theta_m<t_0+T, \\ \zeta_m & \text{if } t_{m+1}=t_0+T, \end{cases}$$

with $\zeta_m = w(t_0+T)-w(t_m)$ being drawn from the distribution of

$$W_{t_0+T-t_m} \quad \text{conditional on} \quad \max_{0\leq s\leq t_0+T-t_m} |W_s| \leq r, \tag{41}$$

where W is an independent standard Brownian motion. For details see Subsection 4.1 below. We so have the following theorem.

Theorem 6 *For the one-step approximation $\overline{V}(t_{m+1})$ due to the exact starting value $\overline{V}(t_m) = V(t_m) = y_m^2$, we have the one step error*

$$\left|\sqrt{V(t_{m+1})} - \sqrt{\overline{V}(t_{m+1})}\right| \leq \left(D_1+\frac{D_2}{V(t_m)}\right) r\left(t_{m+1}-t_m\right). \tag{42}$$

4.1 Simulation of θ_m and ζ_m

For simulating θ_m we utilize the distribution function

$$\mathcal{P}(t) := P(\tau < t),$$

where τ is the first-passage time of the Wiener process $W(t)$ to the boundary of the interval $[-1,1]$. A very accurate approximation $\tilde{\mathcal{P}}(t)$ of $\mathcal{P}(t)$ is the following one:

$$\mathcal{P}(t) \simeq \tilde{\mathcal{P}}(t) = \int_0^t \tilde{\mathcal{P}}'(s)ds \quad \text{with}$$

$$\tilde{\mathcal{P}}'(t) = \begin{cases} \dfrac{2}{\sqrt{2\pi t^3}}(e^{-\dfrac{1}{2t}} - 3e^{-\dfrac{9}{2t}} + 5e^{-\dfrac{25}{2t}}), \ 0 < t \le \dfrac{2}{\pi}, \\ \dfrac{\pi}{2}(e^{-\dfrac{\pi^2 t}{8}} - 3e^{-\dfrac{9\pi^2 t}{8}} + 5e^{-\dfrac{25\pi^2 t}{8}}), \ t > \dfrac{2}{\pi}, \end{cases}$$

and it holds

$$\sup_{t\ge 0}\left|\tilde{\mathcal{P}}'(t) - \mathcal{P}'(t)\right| \le 2.13\times 10^{-16}, \quad \text{and} \quad \sup_{t\ge 0}\left|\tilde{\mathcal{P}}(t) - \mathcal{P}(t)\right| \le 7.04\times 10^{-18},$$

(see for details [16], Ch. 5, Sect. 3 and Appendix A3). Now simulate a random variable U uniformly distributed on $[0,1]$, Then compute $\tau = \mathcal{P}^{-1}(U)$ which is distributed according to $\mathcal{P}$. That is, we have to solve the equation $\tilde{\mathcal{P}}(\tau) = U$, for instance by Newton's method or any other efficient solving routine. Next set $\theta_m = r^2\tau_m$.

For simulating ζ_m in (40) we observe that (41) is equivalent with

$$rW_{r^{-2}(t_0+T-t_m)} \quad \text{conditional on} \quad \max_{0\le u\le r^{-2}(t_0+T-t_m)} |W_u| \le 1.$$

We next sample ϑ from the distribution function $\mathcal{Q}(x; r^{-2}(t_0+T-t_m))$, where $\mathcal{Q}(x;t)$ is the known conditional distribution function (see [16], Ch. 5, Sect. 3)

$$\mathcal{Q}(x;t) := P(W(t) < x \mid \max\nolimits_{0\le s\le t}|W(s)| < 1), \quad -1 \le x \le 1, \tag{43}$$

and set $\zeta_m = r\vartheta$. The simulation of the last step looks rather complicated and may be computationally expensive. However it is possible to take for $w(t_0+T) - w(t_\nu)$ simply any value between $-r$ and r, e.g. zero. This may enlarge the one-step error on the last step but does not influence the convergence order of the elaborated method. Indeed, if we set $w(t_0+T) - w(t_\nu)$ to be zero, for instance, on the last step, we get $\sqrt{\overline{V}(t_0+T)} = y^\nu(t_0+T)$ instead of (40), and

$$\left|\sqrt{V(t_0+T)} - \sqrt{\overline{V}(t_0+T)}\right| \le r\sum_{m=0}^{\nu}\left(D_1 + \frac{D_2}{\overline{V}(t_m)}\right)(t_{m+1} - t_m) + \sigma r, \tag{44}$$

Remark 7 *We have in any step $E\theta_n = r^2$, the random number of steps before reaching t_0+T, say $\nu+1$, is finite with probability one, and $E\nu = O(1/r^2)$. For details see [16], Ch. 5, Lemma 1.5. In a heuristic sense this means that, if we have convergence of order $O(r)$, we obtain accuracy $O(\sqrt{h})$, for an (expected) number of steps $O(1/h)$ similar to the standard Euler scheme.*

5 Convergence theorem

In this section we develop a scheme that generates approximations $\sqrt{\overline{V}(t_0)} = \sqrt{V(t_0)}$, $\sqrt{\overline{V}(t_1)}$, ... , $\sqrt{\overline{V}(t_{n+1})}$, where $n = 0, 1, 2, ...$, and $t_1, ..., t_{n+1}$ are realizations of a sequence of stopping times, and show that the global error in approximation $\sqrt{\overline{V}(t_{n+1})}$ is in fact an aggregated sum of local errors, i.e.,

$$r\sum_{m=0}^{n}\left(D_1 + \frac{D_2}{\overline{V}(t_m)}\right)(t_{m+1} - t_m) \leq rT\left(D_1 + \frac{D_2}{\eta_n^2}\right),$$

with $y_m = \sqrt{\overline{V}(t_m)}$, provided that $y_m \geq \sigma r$ for $m = 0, ..., n$, and so $\eta_n := \min_{0\leq m\leq n} y_m \geq \sigma r$.

Let us now describe an algorithm for the solution of (1) on the interval $[t_0, t_0 + T]$ in the case $\alpha \geq 0$. Suppose we are given $V(t_0)$ and r such that

$$\sqrt{V(t_0)} \geq \sigma r.$$

For the initial step we use the one-step approximation according to the previous section and thus obtain (see (40) and (42))

$$\sqrt{\overline{V}(t_1)} = y^0(t_1) + \frac{\sigma}{2}(w(t_1) - w(t_0)),$$
$$\sqrt{V(t_1)} = \sqrt{\overline{V}(t_1)} + \rho^0(t_1),$$

where

$$\left|\rho^0(t_1)\right| \leq \left(D_1 + \frac{D_2}{V(t_0)}\right) r\,(t_1 - t_0) =: C_0 r(t_1 - t_0). \tag{45}$$

Suppose that

$$\sqrt{\overline{V}(t_1)} \geq \sigma r.$$

We then go to the next step and consider the expression

$$\sqrt{V(t)} = Y(t; t_1) + \frac{\sigma}{2}(w(t) - w(t_1)), \tag{46}$$

where $Y(t; t_1)$ is the solution of the problem (see (34))

$$\frac{dY}{dt} = \frac{\alpha}{Y + \frac{\sigma}{2}(w(t) - w(t_1))} - \frac{k}{2}(Y + \frac{\sigma}{2}(w(t) - w(t_1))), \tag{47}$$
$$Y(t_1; t_1) = \sqrt{V(t_1)},\ t_1 \leq t \leq t_1 + \theta_1.$$

Now, in contrast to the initial step, the value $\sqrt{V(t_1)}$ is unknown and we are forced to use $\sqrt{\overline{V}(t_1)}$ instead. Therefore we introduce $\overline{Y}(t; t_1)$ as the solution of the equation (47) with initial value $\overline{Y}(t_1; t_1) = \sqrt{\overline{V}(t_1)}$. From the previous step we have that

$\left|Y(t_1;t_1)-\overline{Y}(t_1;t_1)\right| = \left|\sqrt{V(t_1)}-\sqrt{\overline{V}(t_1)}\right| = |\rho^0(t_1)| \le C_0 r(t_1-t_0)$. Hence, due to Lemma 2,

$$\left|Y(t;t_1)-\overline{Y}(t;t_1)\right| \le \rho^0(t_1) \le C_0 r(t_1-t_0),\ t_1 \le t \le t_1+\theta_1. \tag{48}$$

Let θ_1 be the first-passage time of the Wiener process $w(t_1+\cdot)-w(t_1)$ to the boundary of the interval $[-r,r]$. If $t_1+\theta_1 < t_0+T$ then set $t_2 := t_1+\theta_1$, else set $t_2 := t_0+T$. In order to approximate $\overline{Y}(t;t_1)$ for $t_1 \le t \le t_2$ let us consider along with equation (47) the equation

$$\frac{dy^1}{dt} = \frac{\alpha}{y^1} - \frac{k}{2}y^1,\ y^1(t_1) = \overline{Y}(t_1;t_1) = \sqrt{\overline{V}(t_1)}.$$

Due to Proposition 3 and Corollary 5 it holds that

$$\left|\overline{Y}(t;t_1)-y^1(t)\right| \le \left(D_1+\frac{D_2}{\overline{V}(t_1)}\right) r\,(t_2-t_1) =: C_1 r\,(t_2-t_1)\,, \quad t_1 \le t \le t_2. \tag{49}$$

and so by (48) we have

$$\left|Y(t;t_1)-y^1(t)\right| \le r(C_0(t_1-t_0)+C_1\,(t_2-t_1)), \quad t_1 \le t \le t_2. \tag{50}$$

We also have (see (46))

$$\sqrt{V(t)} = Y(t;t_1)+\frac{\sigma}{2}(w(t)-w(t_1)) = y^1(t)+\frac{\sigma}{2}(w(t)-w(t_1))+R^1(t), \tag{51}$$

where

$$\left|R^1(t)\right| \le r(C_0(t_1-t_0)+C_1\,(t_2-t_1)),\ t_1 \le t \le t_2. \tag{52}$$

We so define the approximation

$$\sqrt{\overline{V}(t)} := y^1(t)+\frac{\sigma}{2}(w(t)-w(t_1)), \quad \text{that satisfies} \tag{53}$$

$$\sqrt{V(t)} = \sqrt{\overline{V}(t)}+R^1(t)\ , \quad t_1 \le t \le t_2. \tag{54}$$

and then set

$$\sqrt{\overline{V}(t_2)} = y^1(t_2)+\frac{\sigma}{2}(w(t_2)-w(t_1)) = \tag{55}$$

$$y^1(t_2)+\frac{\sigma}{2}\cdot\begin{cases} r\xi_1 & \text{with } P(\xi_1=\pm 1)=1/2, \text{ if } t_2=t_1+\theta_1<t_0+T, \\ \zeta_1 & \text{if } t_2=t_0+T, \end{cases},$$

cf. (40) and (41). We thus end up with a next approximation $\sqrt{\overline{V}(t_2)}$ such that

$$\left|\sqrt{V(t_2)}-\sqrt{\overline{V}(t_2)}\right| = \left|R^1(t_2)\right| \le r(C_0(t_1-t_0)+C_1\,(t_2-t_1)). \tag{56}$$

From the above description it is obvious how to proceed analogously given a generic approximation sequence of approximations $\sqrt{\overline{V}(t_m)}$, $m=0,1,2,...,n$, with $\overline{V}(t_0)=V(t_0)$, that satisfies by assumption

$$
\begin{aligned}
\sqrt{\overline{V}(t_m)} &\geq \sigma r, \quad \text{for } m = 0, ..., n, \quad \text{and} && (57)\\
\left|\sqrt{V(t_n)} - \sqrt{\overline{V}(t_n)}\right| &\leq r \sum_{m=0}^{n-1} \left(D_1 + \frac{D_2}{\overline{V}(t_m)}\right)(t_{m+1} - t_m) && (58)\\
&=: r \sum_{m=0}^{n-1} C_m \left(t_{m+1} - t_m\right).
\end{aligned}
$$

Indeed, consider the expression

$$
\sqrt{V(t)} = Y(t; t_n) + \frac{\sigma}{2}(w(t) - w(t_n)),
$$

where $Y(t; t_n)$ is the solution of the problem

$$
\begin{aligned}
\frac{dY}{dt} &= \frac{\alpha}{Y + \frac{\sigma}{2}(w(t) - w(t_n))} - \frac{k}{2}(Y + \frac{\sigma}{2}(w(t) - w(t_n))), && (59)\\
Y(t_n; t_n) &= \sqrt{V(t_n)}, \ t_n \leq t \leq t_n + \theta_n,
\end{aligned}
$$

for a $\theta_n > 0$ to be determined. Since $\sqrt{V(t_n)}$ is unknown we consider $\overline{Y}(t; t_n)$ as the solution of the equation (59) with initial value $\overline{Y}(t_n; t_n) = \sqrt{\overline{V}(t_n)}$. Due to (58) and Lemma 2 again, we have

$$
\left|Y(t; t_n) - \overline{Y}(t; t_n)\right| \leq r \sum_{m=0}^{n-1} C_m \left(t_{m+1} - t_m\right), \ t_n \leq t \leq t_n + \theta_n.
$$

In order to approximate $\overline{Y}(t; t_n)$ for $t_n \leq t \leq t_n + \theta_n$, we consider the equation

$$
\frac{dy^n}{dt} = \frac{\alpha}{y^n} - \frac{k}{2} y^n, \ y^n(t_n) = \overline{Y}(t_n; t_n) = \sqrt{\overline{V}(t_n)}. \tag{60}
$$

By repeating the procedure (49)-(56) we arrive at

$$
\sqrt{\overline{V}(t)} := y^n(t) + \frac{\sigma}{2}(w(t) - w(t_n)), \ t_n \leq t \leq t_{n+1}, \tag{61}
$$

satisfying

$$
\left|\sqrt{V(t)} - \sqrt{\overline{V}(t)}\right| = |R^n(t)| \leq r \sum_{m=0}^{n} \left(D_1 + \frac{D_2}{\overline{V}(t_m)}\right)(t_{m+1} - t_m), \quad t_n \leq t \leq t_{n+1}, \tag{62}
$$

with

$$
\begin{aligned}
R^n(t) &:= Y(t; t_n) - y^n(t), \quad t_n \leq t \leq t_{n+1}, \text{ and in particular} && (63)\\
\left|\sqrt{V(t_{n+1})} - \sqrt{\overline{V}(t_{n+1})}\right| &\leq r \sum_{m=0}^{n} \left(D_1 + \frac{D_2}{\overline{V}(t_m)}\right)(t_{m+1} - t_m).
\end{aligned}
$$

Remark 8 *In principle it is possible to use the distribution function $\mathcal{Q}$ (see (43)) for constructing $\sqrt{\overline{V}(t)}$ for $t_n < t < t_{n+1}$. However, we rather consider for $t_n \leq t \leq t_{n+1}$ the approximation*

$$\sqrt{\widetilde{V}(t)} := y^n(t) + \frac{\sigma}{2}\widetilde{w}_n(t),\ t_n \leq t \leq t_{n+1},$$

where (a) for $t_{n+1} < t_0 + T$, $\widetilde{w}$ is an arbitrary continuous function satisfying

$$\widetilde{w}(t_n) = 0, \quad \widetilde{w}(t_{n+1}) = w(t_{n+1}) - w(t_n) = r\xi_n, \qquad \max_{t_n \leq t \leq t_{n+1}} |\widetilde{w}_n(t)| \leq r,$$

and (b) for $t_{n+1} = t_0 + T$, one may take $\widetilde{w}(t) \equiv 0$. As a result we get similar to (44) an insignificant increase of the error,

$$\left|\sqrt{V(t)} - \sqrt{\widetilde{V}(t)}\right| \leq r \sum_{m=0}^{n} \left(D_1 + \frac{D_2}{\overline{V}(t_m)}\right)(t_{m+1} - t_m) + \sigma r,\ t_n < t < t_{n+1}.$$

Let us consolidate the above procedure in a concise way.

5.1 Simulation algorithm

- *Set* $\sqrt{\overline{V}(t_0)} = \sqrt{V(t_0)}$.

- *Let the point $(t_n, \sqrt{\overline{V}(t_n)})$ be known for an $n \geq 0$. Simulate independent random variables ξ_n with $P(\xi_n = \pm 1) = 1/2$,* and θ_n as described in subsection 4.1. *If* $t_n + \theta_n < t_0 + T$, set $t_{n+1} = t_n + \theta_n$, else set $t_{n+1} = t_0 + T$.

- *Solve equation (60) on the interval* $[t_n, t_{n+1}]$ with solution y^n and set

$$\sqrt{\overline{V}(t_{n+1})} = y^n(t_{n+1}) + \frac{\sigma}{2} \cdot \begin{cases} r\xi_n & \text{if } t_{n+1} < t_0 + T, \\ 0 & \text{if } t_{n+1} = t_0 + T. \end{cases}$$

So, under the assumption (57) we obtain the estimate (62) (possibly enlarged with a term σr). The next theorem shows that if a trajectory of $V(t)$ under consideration is positive on $[t_0, t_0 + T]$, then the algorithm is convergent on this trajectory. We recall that in the case $2k\lambda \geq \sigma^2$ almost all trajectories are positive, hence in this case the proposed method is almost surely convergent.

Theorem 9 *Let $4k\lambda \geq \sigma^2$ (i.e., $\alpha \geq 0$). Then for any positive trajectory $V(t) > 0$ on $[t_0, t_0 + T]$ the proposed method is convergent on this trajectory. In particular, there exist $\eta > 0$ depending on the trajectory $V(\cdot)$ only, and $r_0 > 0$ depending on η such that*

$$\sqrt{\overline{V}(t_m)} \geq \eta \geq r\sigma, \quad \text{for } m = 0, 1, 2, \ldots$$

for any $r < r_0$. So in particular (57) is fulfilled for all $m = 0, 1, \ldots$, and the estimate (62) implies that for any $r < r_0$,

$$\left|\sqrt{V(t_{n+1})} - \sqrt{\overline{V}(t_{n+1})}\right| \leq r\left(D_1 + \frac{D_2}{\eta^2}\right)T, \quad n = 0, 1, 2, \ldots, \nu.$$

Proof. Let us define

$$\eta := \frac{1}{2} \min_{t_0 \le t \le t_0+T} \sqrt{V(t)} \quad \text{and}$$

$$r_0 := \min \left(\frac{\eta}{\sigma}, \frac{\eta}{\left(D_1 + \frac{D_2}{\eta^2}\right) T} \right), \tag{64}$$

and let $r < r_0$. We then claim that for all m,

$$\sqrt{\overline{V}(t_m)} \ge \eta \ge r\sigma. \tag{65}$$

For $m = 0$ we trivially have

$$\sqrt{\overline{V}(t_0)} = \sqrt{V(t_0)} \ge 2\eta \ge \eta \ge r_0\sigma \ge r\sigma.$$

Now suppose by induction that $\sqrt{\overline{V}(t_j)} \ge \eta$ for $j = 0, ..., m$.Then due to (63) we have

$$\left| \sqrt{V(t_{m+1})} - \sqrt{\overline{V}(t_{m+1})} \right| \le r \left(D_1 + \frac{D_2}{\eta^2} \right) T \le r_0 \left(D_1 + \frac{D_2}{\eta^2} \right) T \le \eta$$

because of (64). Thus, since $\sqrt{V(t_{m+1})} \ge 2\eta$, it follows that $\sqrt{\overline{V}(t_{m+1})} \ge \eta \ge r\sigma$. This proves (65) and the convergence for $r \downarrow 0$. ■

Remark 10 . *In the case where $4k\lambda \ge \sigma^2 > 2k\lambda$ trajectories will reach zero with positive probability, that is convergence on such trajectories is not guaranteed by Theorem 9. So it is important to develop some method for continuing the simulations in cases of very small $\overline{V}(t_m)$. One can propose different procedures, for instance, one can proceed with standard SDE approximation methods relying on some known scheme suitable for small V (e.g. see [3]). However, the uniformity of the simulation would be destroyed in this way. We therefore propose in the next section a uniform simulation method that may be started in a value $\overline{V}(t_m)$ close to zero.*

6 Simulation of trajectories near to zero

Henceforth we assume that $\alpha > 0$. Let us suppose that $\sqrt{\overline{V}(t_n)} = y_n \ge \sigma r$ and consider conditions that guarantee that $\sqrt{\overline{V}(t_{n+1})} \ge \sigma r$ under $t_{n+1} \le t_0 + T$. Of course in the case $\xi_n = 1$ this is trivially fulfilled, and we thus consider the case $\xi_n = -1$, yielding

$$\sqrt{\overline{V}(t_{n+1})} = y^n(t_{n+1}) - \frac{\sigma r}{2} = [y_n^2 e^{-k(t_{n+1}-t_n)} + \frac{2\alpha}{k}(1 - e^{-k(t_{n+1}-t_n)})]^{1/2} - \frac{\sigma r}{2}.$$

We so need

$$y_n^2 e^{-k(t_{n+1}-t_n)} + \frac{2\alpha}{k}(1 - e^{-k(t_{n+1}-t_n)}) \ge \frac{9\sigma^2 r^2}{4}, \quad \text{i.e.}$$

$$\left(y_n^2 - \frac{2\alpha}{k} \right) e^{-k(t_{n+1}-t_n)} \ge \frac{9\sigma^2 r^2}{4} - \frac{2\alpha}{k}. \tag{66}$$

Since we are interested in properties of algorithms when $r \downarrow 0$, we may further assume w.l.o.g. that $9\sigma^2 r^2/4 - 2\alpha/k < 0$, i.e.

$$r < \frac{2}{3}\sqrt{\frac{2\alpha}{k\sigma^2}}. \tag{67}$$

Under assumption (67), (66) is obviously fulfilled when $y_n \geq \sqrt{2\alpha/k}$. If $y_n < \sqrt{2\alpha/k}$ we need

$$e^{-k(t_{n+1}-t_n)} \leq \frac{\frac{9\sigma^2 r^2}{4} - \frac{2\alpha}{k}}{y_n^2 - \frac{2\alpha}{k}}, \qquad \text{hence} \;\; t_{n+1} - t_n \geq \frac{1}{k} \ln \frac{\frac{2\alpha}{k} - y_n^2}{\frac{2\alpha}{k} - \frac{9\sigma^2 r^2}{4}},$$

which is fulfilled if

$$y_n \geq \frac{3}{2}\sigma r. \tag{68}$$

Note that (67) is equivalent with $3\sigma r/2 < \sqrt{2\alpha/k}$, and so (68) is the condition we were looking for. Conversely, if $\sigma r \leq y_n < 3\sigma r/2$, then $\sqrt{\overline{V}(t_{n+1})} < \sigma r$ with positive probability. In view of the above considerations, one may carry out the algorithm of Subsection 5.1 as long as (68) is fulfilled. Let us say that $\mathfrak{n}$ was the last step where (68) was true. Then the aggregated error of $\sqrt{\overline{V}(t_{n+1})}$ due to the algorithm up to step $\mathfrak{n}$ may be estimated by (cf. (57) and (58)),

$$\left|\sqrt{V(t_{\mathfrak{n}+1})} - \sqrt{\overline{V}(t_{\mathfrak{n}+1})}\right| \leq$$

$$r\sum_{m=0}^{\mathfrak{n}} \left(D_1 + \frac{D_2}{y_m^2}\right)(t_{m+1} - t_m) = r\sum_{m=0}^{\mathfrak{n}} \left(D_1 + \frac{D_2}{\overline{V}(t_m)}\right)(t_{m+1} - t_m). \tag{69}$$

Let us recall that our primal goal is a scheme where $\sqrt{V(t)} - \sqrt{\overline{V}(t)} \downarrow 0$, almost surely and uniformly in $t_0 \leq t \leq t_0 + T$. In this respect, and in particular in the case $\sigma^2 > 2k\lambda$ where trajectories may attain zero with positive probability, it is not recommended to carry out scheme 5.1 all the way through until (68) is not satisfied anymore. Indeed, if the trajectory attains zero, the worst case almost sure error bound would then be when all y_m would be close to $3\sigma r/2$, hence of order $O(1/r)$. That is, no convergence on such trajectories. We therefore propose to perform scheme 5.1 up to a (stopping) index $\mathfrak{m}$, defined by

$$\sqrt{\overline{V}(t_k)} \geq \frac{1}{2}Ar^a > \frac{3}{2}\sigma r, \;\; k = 0, 1, ..., \mathfrak{m}, \;\; \text{and} \;\; \sqrt{\overline{V}(t_{\mathfrak{m}+1})} < \frac{1}{2}Ar^a, \tag{70}$$

where A is a positive constant and $0 < a < 1/2$ is to be determined suitably. A pragmatic choice would be $a = 1/3$ (see Remark 11). Due to (70) and (69) with $\mathfrak{n}$ replaced by $\mathfrak{m}$ we then have,

$$\left|\sqrt{V(t_{\mathfrak{m}+1})} - \sqrt{\overline{V}(t_{\mathfrak{m}+1})}\right| \leq r\sum_{m=0}^{\mathfrak{m}} \left(D_1 + \frac{4D_2}{A^2 r^{2a}}\right)(t_{m+1} - t_m) \leq: D_3 r^{1-2a}. \tag{71}$$

for some constant $D_3 > 0$.

Let us now fix a realization $t_{m+1} := t_{\mathfrak{m}+1}$, and consider two solutions of equation (1) starting at the moment t_{m+1} from $\overline{v} := \overline{V}(t_{m+1})$ (known value) and $v = V(t_{m+1})$ (true but unknown value), denoted by $V_{t_{m+1},\overline{v}}$ and $V_{t_{m+1},v}$, respectively, Let ϑ_x, $0 \leq x < A^2 r^{2a}$, be the first time at which the solution $V_{t_{m+1},x}(t + t_{m+1})$ of (1) attains the level $A^2 r^{2a}$, hence

$$0 \leq V_{t_{m+1},x}(t_{m+1} + t) < A^2 r^{2a},\ 0 \leq t < \vartheta_x,\ V_{t_{m+1},x}(t_{m+1} + \vartheta_x) = A^2 r^{2a}.$$

A construction of the distribution function of ϑ_x is worked out in Section 6.1. Let us now denote $t_{m+2} := t_{m+1} + \vartheta$ with $\vartheta = \vartheta_{\overline{v}}$. (For simplicity and w.l.og. we assume that $t_{m+2} < t_0 + T$). We then naturally set

$$\overline{V}(t_{m+2}) = V_{t_{m+1},\overline{v}}(t_{m+1} + \vartheta) = A^2 r^{2a}.$$

The solutions $V_{t_{m+1},\overline{v}}$ and $V_{t_{m+1},v}$ correspond to two solutions $\overline{Y}(t, t_{m+1})$ and $Y(t, t_{m+1})$ of (20) with $\varphi(t) = w(t) - w(t_{m+1})$, $t \geq t_{m+1}$, starting in $\overline{Y}(t_{m+1}, t_{m+1}) = \sqrt{\overline{v}}$ and $Y(t_{m+1}, t_{m+1}) = \sqrt{v}$, respectively. Due to Lemma 2, see Remark 3, and (71) it thus follows that

$$\begin{aligned}\left|\sqrt{V_{t_{m+1},v}(t)} - \sqrt{V_{t_{m+1},\overline{v}}(t)}\right| &= \left|Y(t, t_{m+1}) - \overline{Y}(t, t_{m+1})\right| \leq \left|\sqrt{V(t_{m+1})} - \sqrt{\overline{V}(t_{m+1})}\right| \\ &\leq D_3 r^{1-2a}, \quad t_{m+1} \leq t \leq t_{m+2}, \qquad (72)\end{aligned}$$

and in particular

$$\sqrt{V(t_{m+2})} - \sqrt{\overline{V}(t_{m+2})} \leq D_3 r^{1-2a}.$$

In contrast to the previous steps we now specify the behavior of $\overline{V}(t)$ on $[t_{m+1}, t_{m+2}]$ by

$$\overline{V}(t) = V_{t_{m+1},\overline{v}}(t), \quad t_{m+1} \leq t \leq t_{m+2}, \qquad (73)$$

which we actually do not know. However, we do know that $\overline{V}(t_{m+2}) = V_{t_{m+1},\overline{v}}(t_{m+2}) = A^2 r^{2a}$, and that $\overline{V}$ is bounded on $[t_{m+1}, t_{m+2}]$ by $A^2 r^{2a}$. Therefore, if we just take a straight line $L(t)$ that connects the points $(t_{m+1}, \sqrt{\overline{v}})$ and (t_{m+2}, Ar^a) as an approximation for $\sqrt{\overline{V}(t)}$, then $\sqrt{\overline{V}(t)} - L(t) \leq Ar^a$, $t_{m+1} \leq t \leq t_{m+2}$. By (72) and (73) we then also have

$$\sqrt{V(t)} - L(t) \leq Ar^a,\ t_{m+1} \leq t \leq t_{m+2}.$$

Thus, the accuracy of the approximation to $\sqrt{V}$ for $0 \leq t \leq t_{m+1}$ outside the band $\left(0, \frac{1}{2}Ar^a\right)$ is of order $O(r^{1-2a})$, and for $t_{m+1} < t < t_{m+2}$ inside the band $(0, Ar^a)$ of order $O(r^a)$. But, at the boundary point $\overline{V}(t_{m+2}) = A^2 r^{2a}$ the accuracy is of order $O(r^{1-2a})$ again. Finally, the scheme may be continued from the state

$$\sqrt{\overline{V}(t_{m+2})} = Ar^a$$

with the algorithm of Subsection 5.1.

Remark 11 *From the above construction it is clear that for* $a = 1/3$ *in (70) the accuracy for* $0 \leq t \leq t_{m+1}$ *outside the band* $\left(0, \frac{1}{2}Ar^a\right)$, *and for* $t_{m+1} < t < t_{m+2}$ *inside the band* $(0, Ar^a)$ *are of the same order. However, an exponent* $0 < a < 1/3$ *would give a higher accuracy outside the band* $\left(0, \frac{1}{2}Ar^a\right)$ *and at the exit points of the band* $(0, Ar^a)$, *while inside the band the accuracy is worse but uniformly bounded by* Ar^a.

6.1 Simulation of ϑ_x

In order to carry out the above simulation method for trajectories near zero we have to find the distribution function of $\vartheta_x = \vartheta_{x,l}$, where $\vartheta_{x,l}$ is the first-passage time of the trajectory $X_{0,x}(s)$, to the level l. For this it is more convenient to change notation and to write (1) in the form

$$dX(s) = k(\lambda - X(s))ds + \sigma\sqrt{X}dw(s),\ X(0) = x, \tag{74}$$

where without loss of generality we take the initial time to be $s = 0$. The function

$$u(t, x) := P(\vartheta_{x,l} < t),$$

is the solution of the first boundary value problem of parabolic type ([16], Ch. 5, Sect. 3)

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2 x\frac{\partial^2 u}{\partial x^2} + k(\lambda - x)\frac{\partial u}{\partial x},\ t > 0,\ 0 < x < l, \tag{75}$$

with initial data

$$u(0, x) = 0, \tag{76}$$

and boundary conditions

$$u(t, 0) \text{ is bounded, } u(t, l) = 1. \tag{77}$$

To get homogeneous boundary conditions we introduce $v = u - 1$. The function v then satisfies:

$$\frac{\partial v}{\partial t} = \frac{1}{2}\sigma^2 x\frac{\partial^2 v}{\partial x^2} + k(\lambda - x)\frac{\partial v}{\partial x},\ t > 0,\ 0 < x < l, \tag{78}$$

$$v(0, x) = -1;\ v(t, 0) \text{ is bounded, } v(t, l) = 1. \tag{79}$$

The problem (78)-(79) can be solved by the method of separation of variables. In this way the Sturm-Liouville problem for the confluent hypergeometric equation (the Kummer equation) arises . This problem is rather complicated however. Below we are going to solve an easier problem as a good approximation to (78)-(79). Along with (74), let us consider the equations

$$dX^+(s) = k\lambda ds + \sigma\sqrt{X^+}dw(s),\ X^+(0) = x, \tag{80}$$

$$dX^-(s) = k(\lambda - l)ds + \sigma\sqrt{X^-}dw(s),\ X^-(0) = x, \tag{81}$$

with $0 \le l < \lambda$. It is not difficult to prove the following inequalities

$$X^-(s) \le X(s) \le X^+(s). \tag{82}$$

According to (82), we consider three boundary value problems: first (75)-(77) and next similar ones for the equations

$$\begin{aligned}\frac{\partial u^+}{\partial t} &= \frac{1}{2}\sigma^2 x\frac{\partial^2 u^+}{\partial x^2} + k\lambda\frac{\partial u^+}{\partial x},\ t > 0,\ 0 < x < l,\\ \frac{\partial u^-}{\partial t} &= \frac{1}{2}\sigma^2 x\frac{\partial^2 u^-}{\partial x^2} + k(\lambda - l)\frac{\partial u^-}{\partial x},\ t > 0,\ 0 < x < l.\end{aligned} \tag{83}$$

From (82) it follows that

$$u^-(t, x) \le u(t, x) \le u^+(t, x),$$

hence

$$v^-(t,x) \le v(t,x) \le v^+(t,x),$$

where $v^- = u^- - 1,\ v^+ = u^+ - 1$.

As the band $0 < x < l = A^2 r^{2a}$, for a certain $a > 0$, is narrow due to small enough r, the difference $v^+ - v^-$ will be small and so we can consider the following problem

$$\frac{\partial v^+}{\partial t} = \frac{1}{2}\sigma^2 x \frac{\partial^2 v^+}{\partial x^2} + k\lambda \frac{\partial v^+}{\partial x},\ t > 0,\ 0 < x < l, \tag{84}$$

$$v^+(0,x) = -1;\ v^+(t,0) \text{ is bounded, } v^+(t,l) = 0, \tag{85}$$

as a good approximation of (78)-(79). Henceforth we write $v := v^+$. By separation of variables we get as elementary independent solutions to (84), $\mathcal{T}(t)\mathcal{X}(x)$, where

$$\mathcal{T}'(t) + \mu\mathcal{T}(t) = 0, \quad \text{i.e.} \quad \mathcal{T}(t) = \mathcal{T}_0 e^{-\mu t}, \quad \mu > 0, \quad \text{and} \tag{86}$$

$$\frac{1}{2}\sigma^2 x \mathcal{X}'' + k\lambda\mathcal{X}' + \mu\mathcal{X} = 0, \quad \mathcal{X}(0+) \text{ is bounded, } \mathcal{X}(l) = 0. \tag{87}$$

It can be verified straightforwardly that the solution of (87) can be obtained in terms of Bessel functions of the first kind (e.g. see [4]),

$$\mathcal{X}(x) = \mathcal{X}_\gamma^\pm(x) := x^\gamma J_{\pm 2\gamma}\left(\sigma^{-1}\sqrt{8\mu x}\right) = x^\gamma O(x^{\pm\gamma}) \quad \text{if} \quad x \downarrow 0,$$

with

$$\gamma := \frac{1}{2} - \frac{k\lambda}{\sigma^2}. \tag{88}$$

Since $\mathcal{X}(x)$ has to be bounded for $x \downarrow 0$ we may take (regardless the sign of γ (!))

$$\mathcal{X}(x) = \mathcal{X}_\gamma^-(x) =: \mathcal{X}_\gamma(x) = x^\gamma J_{-2\gamma}\left(\sigma^{-1}\sqrt{8\mu x}\right). \tag{89}$$

In our setting we have $\alpha > 0$, i.e. $\gamma < 1/4$.

The following derivation of a Fourier-Bessel series for v is standard but included for convenience of the reader. Denote the positive zeros of J_ν by $\pi_{\nu,m}$, for example,

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad \text{and} \quad \pi_{1/2,m} = m\pi, \quad m = 1, 2, ... \tag{90}$$

Then the (homogeneous) boundary condition $\mathcal{X}_\gamma(l) = 0$ yields

$$\sigma^{-1}\sqrt{8\mu l} = \pi_{-2\gamma,m}, \quad \text{i.e.,} \quad \mu_m := \frac{\sigma^2\pi^2_{-2\gamma,m}}{8l} \tag{91}$$

and we have

$$\mathcal{X}_{\gamma,m}(x) := x^\gamma J_{-2\gamma}\left(\sigma^{-1}\sqrt{8\mu_m x}\right) = x^\gamma J_{-2\gamma}\left(\pi_{-2\gamma,m}\sqrt{\frac{x}{l}}\right).$$

By the well-known orthogonality relation

$$\int_0^1 zJ_{-2\gamma}(\pi_{-2\gamma,k}z)J_{-2\gamma}(\pi_{-2\gamma,k'}z)dz = \frac{\delta_{k,k'}}{2}J^2_{-2\gamma+1}(\pi_{-2\gamma,k}),$$

we get by setting $z = \sqrt{x/l}$

$$\int_0^l J_{-2\gamma}(\pi_{-2\gamma,m}\sqrt{\frac{x}{l}})J_{-2\gamma}(\pi_{-2\gamma,m'}\sqrt{\frac{x}{l}})dx = l\delta_{m,m'}J^2_{-2\gamma+1}(\pi_{-2\gamma,m}), \quad \text{hence}$$

$$\int_0^l \mathcal{X}_{\gamma,m}(x)\mathcal{X}_{\gamma,m'}(x)x^{-2\gamma}dx = l\delta_{m,m'}J^2_{-2\gamma+1}(\pi_{-2\gamma,m}).$$

Now set

$$v(t,x) = \sum_{m=1}^{\infty} \beta_m e^{-\mu_m t}\mathcal{X}_{\gamma,m}(x), \qquad 0 \le x \le l. \tag{92}$$

For $t = 0$ we have due to the initial condition $v(0,x) = -1$,

$$-1 = \sum_{m=1}^{\infty} \beta_m \mathcal{X}_{\gamma,m}(x).$$

So for any $p = 1, 2, ...,$

$$-\int_0^l \mathcal{X}_{\gamma,p}(x)x^{-2\gamma}dx = \beta_p l J^2_{-2\gamma+1}(\pi_{-2\gamma_\kappa,p}), \quad \text{i.e.}$$

$$\beta_p = -\frac{\int_0^l \mathcal{X}_{\gamma,p}(x)x^{-2\gamma}dx}{l J^2_{-2\gamma+1}(\pi_{-2\gamma,p})}. \tag{93}$$

Further it holds that

$$\begin{aligned}\int_0^l \mathcal{X}_{\gamma,p}(x)x^{-2\gamma}dx &= \int_0^l x^{-\gamma}J_{-2\gamma}\left(\pi_{-2\gamma,p}\sqrt{\frac{x}{l}}\right)dx \\ &= 2l^{-\gamma+1}\int_0^1 z^{-2\gamma+1}J_{-2\gamma}\left(\pi_{-2\gamma,p}z\right)dz \\ &= 2l^{-\gamma+1}\frac{J_{-2\gamma+1}\left(\pi_{-2\gamma,p}\right)}{\pi_{-2\gamma,p}}\end{aligned}$$

by well-known identities for Bessel functions (e.g. see [4]), and (93) thus becomes

$$\beta_p = -\frac{2}{l^\gamma \pi_{-2\gamma,p}J_{-2\gamma+1}(\pi_{-2\gamma,p})}, \qquad p = 1, 2, ... \tag{94}$$

So, from $v = u - 1$, (86) (89), (91), (94), and (92) we finally obtain

$$u(t,x) = 1 - 2x^\gamma l^{-\gamma}\sum_{m=1}^{\infty}\frac{J_{-2\gamma}\left(\pi_{-2\gamma,m}\sqrt{\frac{x}{l}}\right)}{\pi_{-2\gamma,m}J_{-2\gamma+1}(\pi_{-2\gamma,m})}\exp\left[-\frac{\sigma^2\pi^2_{-2\gamma,m}}{8l}t\right], \quad 0 \le x \le l. \tag{95}$$

Example 12 *For $\gamma = -1/4$ we get from (95) by (90) straightforwardly,*

$$u(t,x) = 1 + \frac{2}{\pi}\sqrt{\frac{l}{x}}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\sin\left(\pi m\sqrt{\frac{x}{l}}\right)\exp\left[-\frac{\sigma^2\pi^2 m^2}{8l}t\right].$$

For solving (83) we set $\lambda^- := \lambda - l$, and then apply the Fourier-Bessel series (95) with γ replaced by

$$\gamma^- := \frac{1}{2} - \frac{k\lambda^-}{\sigma^2} = \gamma + \frac{kl}{\sigma^2}. \tag{96}$$

Example 13 *We now consider some numerical examples concerning $u^+ = u$ in (95) and u^- given by (95) due to (96). Note that actually in (95) the function u only depends on σ, l, and γ. That is, u depends on σ, l, and the product $k\lambda$. Let us consider a CIR process with $\sigma = 1$, $\lambda = 1$, $k = 0.75$, and let us take $l = 0.1$. We then compare u^+, which is given by (95) for $\gamma = -0.25$ due to (88) (see Example 12), with u^- given by (95) for $\gamma^- = -0.175$ due to (96). The results are depicted in Figure 1. The sums corresponding to (95) are computed with five terms (more terms did not give any improvement).*

Normalization of $u(t,x)$

For practical applications it is useful to normalize (95) in the following way. Let us treat γ as essential but fixed parameter, introduce as new parameters

$$\frac{x}{l} = \widetilde{x}, \quad 0 < \widetilde{x} \leq 1, \qquad \frac{\sigma^2 t}{8l} = \widetilde{t}, \quad \widetilde{t} \geq 0,$$

and consider the function

$$\widetilde{u}(\widetilde{t}, \widetilde{x}) := 1 - 2\widetilde{x}^{\gamma} \sum_{m=1}^{\infty} \frac{J_{-2\gamma}\left(\pi_{-2\gamma,m}\sqrt{\widetilde{x}}\right)}{\pi_{-2\gamma,m} J_{-2\gamma+1}(\pi_{-2\gamma,m})} \exp\left[-\pi^2_{-2\gamma,m}\widetilde{t}\right], \quad 0 < \widetilde{x} \leq 1, \quad \widetilde{t} \geq 0,$$

that is connected to (95) via

$$\widetilde{u}(\widetilde{t}, \widetilde{x}) = \widetilde{u}(\frac{\sigma^2 t}{8l}, \frac{x}{l}) = u(\frac{8l\widetilde{t}}{\sigma^2}, l\widetilde{x}).$$

For simulation of ϑ_x we need to solve the equation

$$u(\vartheta_x, x) = U, \quad \text{where } U \sim \text{Uniform}[0,1].$$

For this we set $\widetilde{x} = x/l$ and solve the normalized equation $\widetilde{u}(\widetilde{\vartheta}_{\widetilde{x}}, \widetilde{x}) = U$, and then take

$$\vartheta_x = \frac{8l}{\sigma^2}\widetilde{\vartheta}_{\widetilde{x}}.$$

Note that

$$P(\vartheta_x < t) = P(\widetilde{\vartheta}_{\widetilde{x}} < \frac{\sigma^2 t}{8l}) = \widetilde{u}(\frac{\sigma^2 t}{8l}, \frac{x}{l}).$$

We have plotted in Figure 2 the normalized function $\widetilde{u}(\widetilde{t}, \widetilde{x})$ for $\gamma = -1/4$.

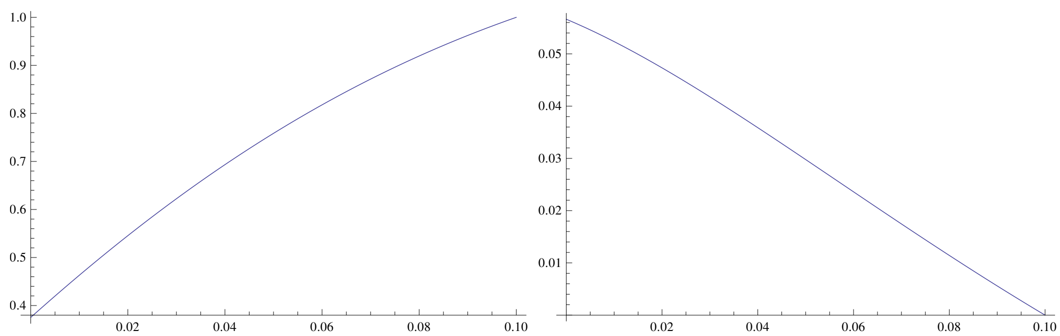

Figure 1: Upper panel $u^+(0.1, x)$, lower panel $u^+(0.1, x) - u^-(0.1, x)$, for $0 \leq x \leq 0.1$

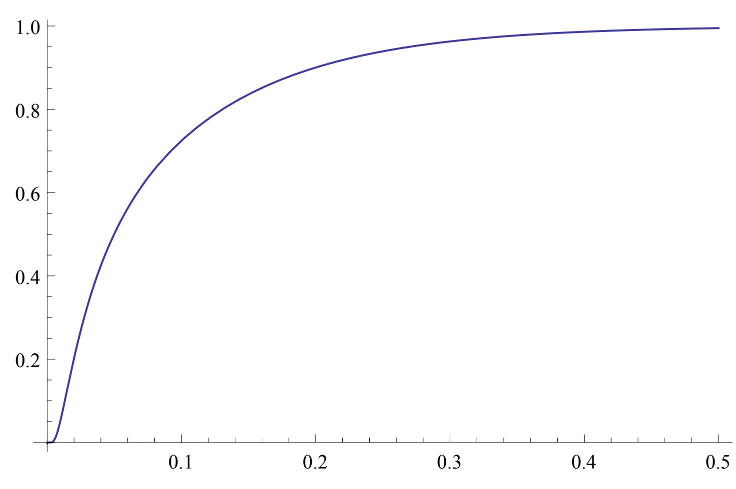

Figure 2: Normalized distribution function $\widetilde{u}(\widetilde{t}, \widetilde{x})$ for $\gamma = -1/4$

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